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Quantifying entropy and asymmetry in convective and magnetic turbulence

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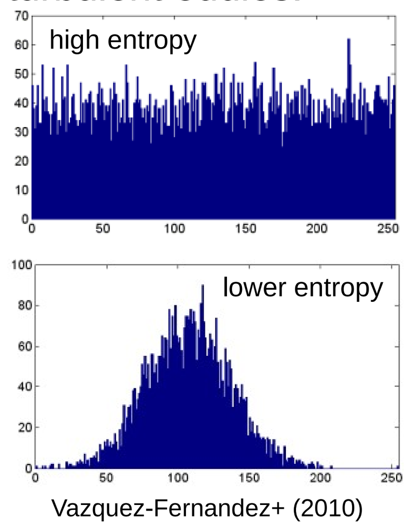
Information entropy measures the disorder or inherent unpredictability of spatial and temporal structures in a time-series or spatially in a high-dimensional dynamical system. Statistical complexity characterises disequilibrium (even given a fixed entropy), and can distinguish deterministic from stochastic physics (chaos vs noise). Related measures of causality quantify the relative influence of time-irreversible and -reversible processes, or spatial asymmetry, handedness and directionality. Calculating these entropic scores from output from simulations can measure strengths of coherent structures or signify turbulent transitions. The entropic cost of numerical approximation schemes (e.g. generalised quasilinear models) is objectively derivable by comparison to DNS. We consider diverse applications to (e.g.) fluid thermal convection, onset of turbulence in magneto-rotational instability, and gyrokinetic plasma turbulence.

Information entropy = H

- Evolving systems show a distribution of **power in modes** or **probabilities of states**, described by **PDF**. Any conceivable system: from rolling dice to turbulent eddies.
- Given a PDF of states, how many **binary questions** do we need to identify the **present** state, in the **"game of twenty questions"**? $-\log_2(p_i)$
- **Information entropy** is the mean: $$S = -\sum_i p_i \log_2(p_i)$$ $$H \equiv S/S_{\max}$$
- A **uniform** PDF gives max entropy. $S_{\max} = \log_2 N$
- **FLAW:** H doesn't change if we shuffle the histogram bins. This "entropy" definition is blind to "**order**."

(e.g. Shannon 1949; Powell & Percival 1979; Xi & Gunton 1995; Miranda+ 2015)

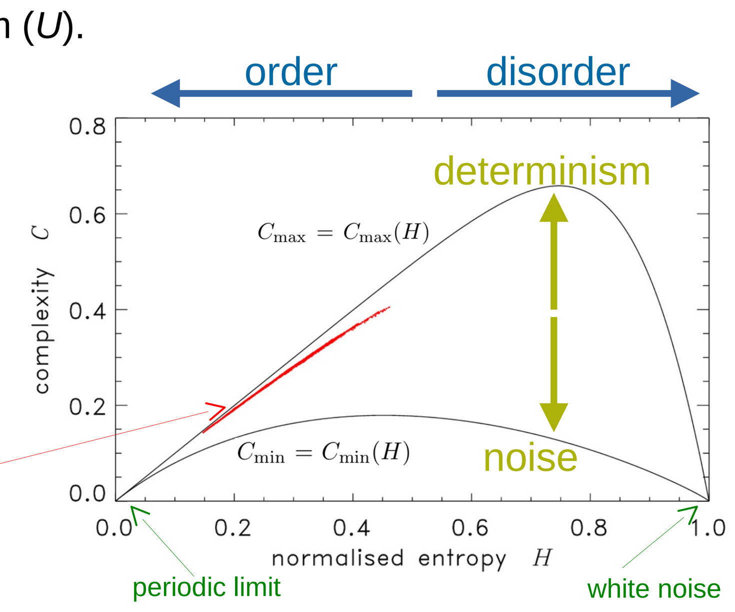


Vazquez-Fernandez+ (2010)

Statistical complexity = C

- Measure **divergence** of PDF (P) from equilibrium (U). $$J(P,U) = S\left(\frac{P+U}{2}\right) - \frac{S(P)}{2} - \frac{S(U)}{2}$$
- Normalise disequilibrium: $Q = J / J_{\max}$
- **Complexity** = disequilibrium × unpredictability $$C \equiv QH$$ (Martin+ 2006)
  - High $C \Rightarrow$ deterministic / fractals / **chaos**;
  - Low $C \Rightarrow$ stochastic / simple / **noise**.

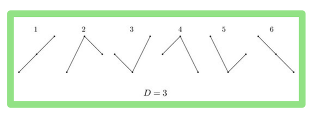
e.g. scatter-plot from gyrokinetic simulations of ion temperature gradient (ITG) driven turbulence

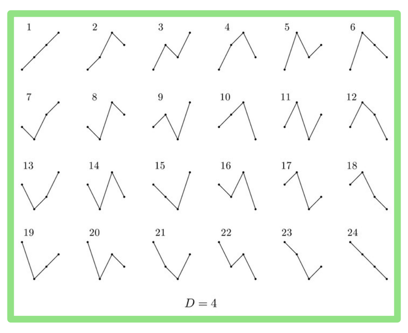


Ordinal statistics

- Given a **time series** (or spatial pattern), $$y = \{y_0, y_1, \ldots, y_{N-1}\}$$
- Select **subseries** s_n of size D and lag τ. $$s_n = \{y_n, y_{n+\tau}, \ldots, y_{n+(D-1)\tau}\}$$
- What is the **rank order** of elements? $$\pi_n = \mathrm{sort}(s_n)$$
- Count PDF(π_n) of **pattern** frequencies.
- Find **ordinal entropy** and **complexity**.
- Results encode **order** but not **power**!
- **FLAW:** tunable values D, τ ∴ not universal.

(see: Bandt & Pompe 2002; Zunino+ 2017)





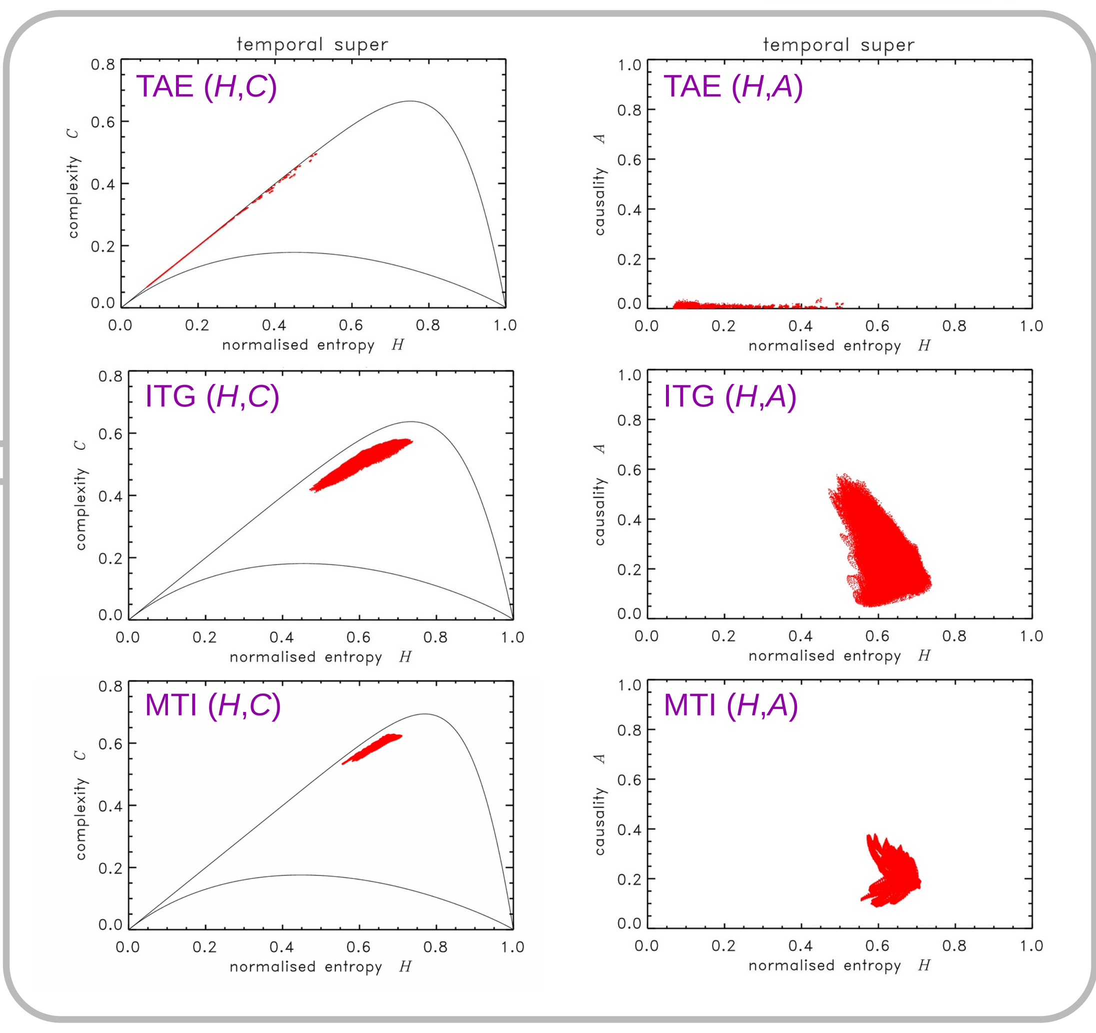
Asymmetry or causality = A

- Each pattern π has a **reversed** partner π'.
- Divergence of **actual** PDF from any **expected** distribution W is…. $$\mathscr{D}(P,W) \equiv \sum_i p_i \log_2(p_i/w_i)$$ (Kullback & Leibler 1951)
- Hypothesis: space- or time-reversible pattern PDFs? $$M = (P + P')/2$$
- Measure spatial **directionality** or time **irreversibility** or **intrinsic causality**. $$A = \frac{1}{2}\mathscr{D}(P,M) + \frac{1}{2}\mathscr{D}(P',M)$$

e.g. Martinez+ (2018, 2023)

Fourier × ordinal statistics? ✓

- Obtain the Fourier **power** a_i at frequencies f_i
- Reample time-series at intervals ~$1/f_i$ again separately for each i.
- Count **pattern** frequencies π_j and normalise by total power a_i.
- 2-parameter PDF(f_i,π_j) gives a more universal **super (H, C, A)**



Gyrokinetic turbulence

Plasma particles in a **tokamak** fusion reactor gyrate tightly around magnetic field lines. Fluctuating phase-space densities and electromagnetic fields mediate diverse forms of **turbulence**. We illustrate entropic properties of the || magnetic potential, in **GENE gyrokinetic** simulations of a local **flux tube** box. **RIGHT:** scatter-plots of (H,C) and (H,A) with dots calculated from time evolution at given spatial points. Plasma phenomena fall in different regions:

- **TAE** = "toroidal Alfven eigenmodes" are periodic patterns resembling acoustic modes. Temporal variability is highly deterministic (C≈H) and time-reversible (A≈0).
- **ITG** = "ion temperature gradient" driven turbulence. Temporal entropy and complexity are high in the chaotic range, but variability is time-irreversible up to A≈0.6.
- **MTI** = "microtearing instability", where some magnetic flux surfaces connect to their own tails, affecting current distributions, driving turbulence. Highish H and C imply chaos but low A implies (almost) time-reversibility.

(see: Beer+ 1995; Goerler 2009; Ajay C.J. 2023; further simulations in progress ….)

Magnetorotational instability

- Astrophysical **accretion discs** need effective viscosity to feed radial **mass inflow**. **MRI = magnetorotational instability:** orbital shear winds up $\boldsymbol{B}$; and magnetic torques drive turbulence (Velikhov 1959; Balbus & Hawley 1991)
- MRI can occur in **magnetic Taylor-Couette** experiments with conductive fluid sheared between **rotating cylinders**, $r_i < r < r_o$ (e.g. Hollerbach & Rüdiger 2005; Hung+ 2019)
- Simulations find various oscillatory states and spatial modes near turbulent onset. We reanalyse five cases for entropy/complexity (Guseva+2017; Guseva & Tobias 2023).
- vary field strength $\mathcal{Ha} \equiv B_0 (r_o\text{-}r_i) / (\sigma/\rho\nu)^{1/2}$ → regimes of torque fluctuations:

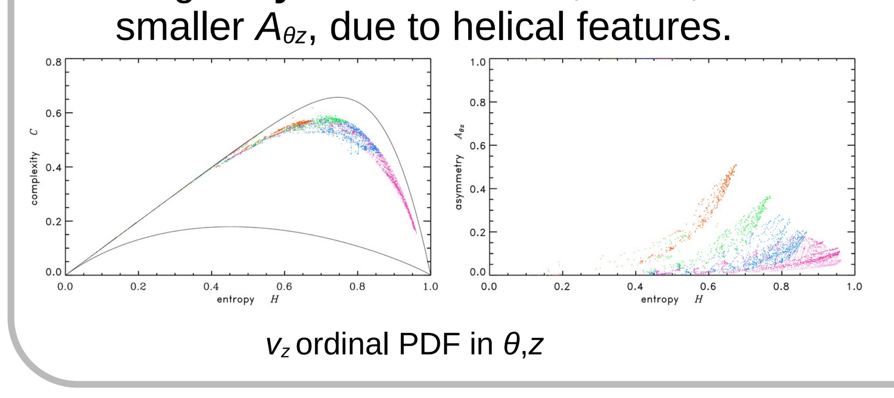
$\mathcal{Ha}$=100 v.chaotic → $\mathcal{Ha}$=120 chaotic → $\mathcal{Ha}$=140 two periods → $\mathcal{Ha}$=145 one period → $\mathcal{Ha}$=149 standing wave

Spatial variability (H,C,A):

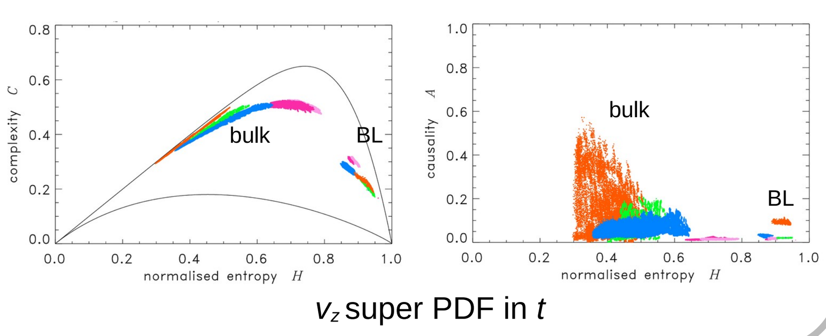
- Fourier 0.2≲H≲0.4 for chaotic cases; 0.1≲H≲0.2 for periodic cases. Deterministic, C≳0.98H.
- Ordinal entropies span a swathe depending on $\mathcal{Ha}$. and scales τ_θ, τ_z.
- Highish $C \Rightarrow$ mainly deterministic.
- Large **asymmetries in** A_θ, A_z≲1; but smaller $A_{\theta z}$, due to helical features.

Temporal variability (H,C,A):

- States occur at distinct (H,C).
- Chaos is impure with **noise**.
- B_θ, v_θ are trivial, C≈H≈0.4.
- v_r, v_z **bimodal** due to boundary noise.
- **Irreversibility/causality** is *stronger* in periodic cases: A≲0.04, 0.04, 0.25, 0.53, 0.52.



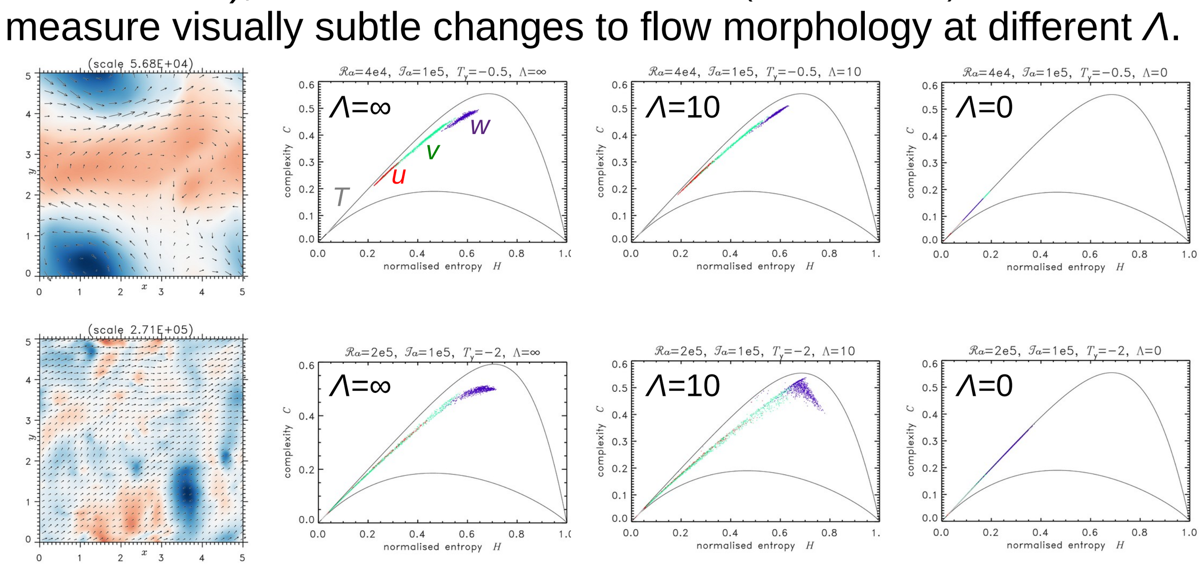
v_z ordinal PDF in θ,z



v_z super PDF in t

Rotating thermal convection

- **Rayleigh–Bénard convection** between hot/cold surfaces, with 45° global **rotation**, like a local box of atmosphere. (Hathaway & Somerville 1986; Currie 2014)
- Vary: $\mathcal{Ra}$ = Rayleigh number, T_y = meridional thermal gradient
- **GQL** = **generalised quasilinear approximation** divides (k_x,k_y) space into "low" and "high" modes (background vs fluctuations), at a wavenumber cutoff Λ. (Saxton+ 2023)
- **Fourier spatial** (H,C) measure visually subtle changes to flow morphology at different Λ.
  - **($\mathcal{Ra}$,T_y)=(4x10[4],-0.5) forms a giant vortex.** Distributions of u,v,w,T straighten and raise C, for Λ=10. Λ=0 locks everything onto $C_{\max}$ curve at low H (quasiperiodic).
  - **($\mathcal{Ra}$,T_y)=(2x10[5],-2) bursting cyclic winds.** Λ=10 raises H and cuts C complexity. Λ=0 locks everything into spatial QPO.



Summary

Analysing spatial and temporal properties of turbulent dynamical systems:

- H **information** entropy measures **disorder**/unpredictability;
- C **complexity** measures disequilibrium; **chaos** vs noise;
- A **asymmetries** measure temporal **causality** or spatial **handedness**.
- We can combine power and ordinal patterns for more informative diagnostics.

In preliminary applications:

- **Plasma turbulence** processes occupy different (H,C,A) regions.
- Astrophysical **MRI turbulence** onset shows shifts in (H,C,A).
- **General quasilinear** treatments of convection subtly distort complexity before entropy.

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